PROFESSOR: Hi. Our main aim today is to successfully conclude block three, and to set the stage for introducing block four. Now by way of brief review, we may have become so involved with our discussion of the last few lessons where we've done the chain rule, that we may have forgotten that the basic ingredient that we had arrived at that set up our proof of the chain rule-- and then after that, we got involved in the computation of how to use the chain rule. But the basic ingredient was the idea that we had obtained a linear approximation, a delta w tan idea. And this concept leads to the idea of a differential, which is an extension of the idea of differentials as we knew them in part one of our course.

And what I would like to do is to talk somewhat briefly about this topic, to show how it introduces what will be the main body of material in block four and an important application that we can handle now that will show why it's worth understanding certain things about differentials, even on a very elementary level.

At any rate, the topic that I've chosen for today is called "Exact Differentials." And the idea, again, is completely analogous to what we did in calculus of a single variable. We start with $w = f(x, y)$. And we show that if w was a continuously differentiable function of x and y, it made sense to emphasize the expression called delta w tan, which was the partial of f with respect to x evaluated at a given point times delta x, plus the partial of f with respect to y at that same point times delta y.

And what we do is the same thing that we did, as I say, in calculus of a single variable. We introduce a new language. We write delta w tan now as dw. We write delta x as dx, delta y as dy. And what we say is that dw is the total differential of w. To see how this is analogous to what happened in calculus of a single variable, notice, for the sake of argument, that if f happens to be a function of x alone-- so that there is no second variable. In other words, if f is independent of y, notice that the partial of f with respect to y will be 0. And we will then have that w is a function of x alone.

dw, then, will be what? Well in that case, the partial of f with respect to x is the ordinary derivative of f with respect to x times dx. And we're back to our old recipe that dw is dw dx times dx. That this is a natural extension of what happened in calculus of a single variable. All right?

Now what leads into the next block and why we're going to drop this for the time being but to go up to a different aspect of this, is notice that if we can assume that dw can replace delta w-- if we can assume, as we talked about in the case of continuously differentiable functions that the difference between delta w and delta w tan is negligible for sufficiently small delta x and delta y, notice that the equation dw equals f sub x dx plus f sub y dy, since f sub x and f sub y are computed at a given point-- these become what? Linear combinations of dx and dy, a constant times dx plus a constant times dy.

And we're into the subject of linear equations, which we will talk about next time. For the time being, all I want to point out is that dw is called the total differential of w. And that conversely, starting with any expression of the form some function M(x, y) times dx plus some function N(x, y) times dy-- in other words, M(x, y) dx plus N(x, y) dy, no matter what functions of x and y, M and N are, any expression of this form is called a differential.

By the way, you may recall that you were used to calling dx a differential back in calculus of a single variable. Notice that in particular, one choice of capital N is to have N(x, y) be identically 0. We could have M(x, y) be identically 1. In which case, M dx plus N dy would simply be 1 dx plus 0 dy or dx. In other words, notice that this extended definition of a differential for two or more variables includes the definition for a single variable. At any rate, any expression of this type is called a differential.

And again, a very natural question that comes up is who wants differentials? Why do we need them? And perhaps the best application comes from the clue that a certain subject is called "Differential Equations," rather than derivative equations.

Let me illustrate this in terms of what I think is a rather nice example in the sense that it will be hard to guess the answer, but rather easy to visualize what the problem says, at least. Let's imagine that I have a certain curve s in the xy-plane. I don't tell you anything else about that curve except that its slope at any point (x, y) is given by the very interesting relationship that it's the quotient of the square of its distance from the origin over twice the product of its coordinates.

In other words, to find the slope of the curve s at the point (x, y), I square the distance of the point from the origin, take the negative of that, divide by twice the product of the coordinates, and that's going to be the slope. Now certainly that makes sense geometrically. But what type of a curve would have that property? And even assuming that we knew such a curve, how in the world can you express it in a convenient manner?

Well here's the idea. Again, back to the same technique that we used in calculus of a single variable. Algebraically, this is our equation. If we cross-multiply and collect all our terms onto one side of the equation, we obtain what we call the differential equation, see? It's an equation involving a differential. In fact, in this case, using our general notation, M(x, y) is x squared plus y squared. And N(x, y) is 2 xy. We have a differential equal to 0.

Now without going into this right now, suppose I just happen to have a very quick eye. Well in fact, before I even say that, let me point this out. We've had problems like this in part one of our course. The main difference was that we could separate out the variables. We could get the x's and the y's separated. Remember now, y is implicitly a function of x here in this expression because of the equation. See? x and y are no longer independent when we equate this to 0.

The question is though, we cannot separate the x's and the y's here. But if we were lucky-- that's where luck comes in-- we would recognize that treating y as a function of x, the differential on the left-hand side is precisely the differential of 1/3 x cubed plus x squared y. In other words, the partial of this with respect to x is simply x squared-- do I have... I think I may have this thing in reverse. Let me make sure. If I take the partial of this with respect to x, I have 1/3x cubed plus 2xy, right? And if I take the partial with respect to y, I simply have x squared.

At any rate-- what I'm saying is, if we knew a function whose-- see, I have this in reverse. That's what's bothering me here. I think the square should go this way. That's not important. All I'm saying is if I've done this thing correctly, if you take the partial of this with respect to y-- with respect x, you should get this. If you take the partial of this with respect to y, you should get this. And that comes out correctly now. In other words, the partial of this with respect to x is x squared plus y squared. The partial of this with respect to y-- this term drops out, and there's just 2xy.

I'm sorry for that little mistake. But I won't really apologize for it in the sense that once we've found it-- or even if we didn't find it, the key theory is the same. The point is that if we knew a function whose differential was this, then the fact that this differential is 0 means the expression itself must be a constant. In other words, 1/3x cubed plus xy squared must be a constant. And therefore, what we're saying is if we now solve the equation-- I really am sorry to have botched this for you. But again, as I say, the idea basically comes through unimpeded. I now solve for y in terms of x, and I get what? That y is equal to-- transposing here-- plus or minus the square root of c minus 1/3x cubed, all over x.

In other words, the family of curves that satisfies this differential equation is precisely this family here, whatever that may look like. In other words, this is the family of curves that has the interesting slope property that we just discussed.

Now you see, again, the reason I say that I wasn't too concerned about what this function actually was is notice that I was just pulling it out of the hat for you here anyway. I said suppose we could find a function w such that dw was this. The major question that comes up in how one uses differentials and the like is as follows. Suppose all I told you was that we had the differential x squared plus y squared dx plus 2 xy dy, and I said to you, find a function w such that dw will be x squared plus y squared dx plus 2 xy dy.

The key computational aid in doing this involves the fact that if u and v are independent variables-- and it's crucial that they be independent-- that if au plus bv equals cu plus dv, it must be that a equals c, and b equals d. In other words, the only way that two linear combinations of independent variables can be equal is if they're equal coefficient by coefficient.

You see, the proof is very simple. Namely, after all, if u and v are independent variables, why can't I pick v to be 0, let u be any non-0 number. Then if v is 0, when I equate these two, it says au equals cu. Since u is not 0, I can cancel, and obtain that a equals c. And in a similar way letting u equal 0, I have bv equals dv. I can cancel the v's and get that b equals d.

But it's crucial that u and v be independent. Because if u and v are not independent, how do I know that I can have u equal to 0 when v is not 0? For example, let's suppose that v and u were dependent. Suppose, for example, that v equals twice u. Observe, in this case, that 9u plus 4v-- since v is 2u-- turns out to be 17u. On the other hand, 7u plus 5v is 7u plus 10u, which is also 17u. Notice that 9u plus 4v equals 7u plus 5v, even though the coefficients don't line up as far as being equal is concerned. You see?

In other words, notice that being able to equate things coefficient by coefficient hinges on the fact that the variables that we're dealing with are independent. And the key thing going back to this problem is, what were dx and dy? They were delta x and delta y. And as long as x and y are independent variables, certainly the change in x can be done independently of the change in y. In other words, dx and dy are independent variables.

So if we now say, let's find that function w such that dw is x squared plus y squared dx plus 2 xy dy, the one thing we know about dw if it exists, by our basic definition of this section of this lecture, is that dw is the partial of f with respect to x times dx plus the partial of f with respect to y times dy. Since dx and dy are independent variables, the only way these two expressions can be identical is coefficient by coefficient. That means in particular, therefore, that the partial of f with respect to x, which is the coefficient of dx here, must be x squared plus y squared, which is the coefficient of dx here. And similarly, the partial of f with respect to y must equal 2 xy.

Now armed with this information, I can actually go out and construct f. And the way I do this is remember-- as soon as I see the partial of f with respect to x, it means I'm treating y as a constant. If I'm treating y as a constant, I just integrate this thing as if x were the only variable. Treating x as a variable and y as a constant, the integral of x squared is 1/3x cubed. The integral of y squared is simply y squared x. Because y is being treated as a constant. And finally, there must be a constant of integration, but my constant now is any function of y.

In other words, if I have any function of y, it's partial with respect to x, by definition of x and y being independent. The partial of any function of y with respect to x is 0. Therefore my constant of integration in this case is a function of y alone. Again, to look at this in a different perspective, what I'm saying is the most general function of two variables in the whole world whose derivative with respect to x is x squared plus y squared is 1/3x cubed plus y squared x plus a function of y alone.

Now all I don't know so far is what specific function of y is g? In other words, I've determined f now up to this particular arbitrary constant of integration. How do I get a hold of this thing? Notice that as yet, I have not used a piece of information that tells me the partial of f with respect to y is 2 xy. And to utilize that piece of information, what I do is I come to this equation, and I say look at it. Let me, from here, just differentiate this with respect to y, treating x as a constant. If I do that, this term drops out. This term becomes 2 xy. And this, being a function of y alone, its derivative with respect to y is just g prime of y.

So whatever f is, the derivative of f with respect to y is 2 xy plus g prime of y. On the other hand, we also know that the partial of f with respect to y is 2 xy. Consequently, since these are two different expressions for the same quantity, they must be equal. Equating these two, the 2 xy's cancel. We obtain that g prime of y is 0, which means, in this case, that g(y) was a bona fide constant, that not only is g independent of x, it's also independent of y. And we see that the most general function f which has the desired property is what?

All we have to do is now go back to this expression here, which gave us the answer up to g(y), put this value of g(y) in, and we see that the most general function whose total differential is x squared plus y squared dx plus 2 xy dy is 1/3x cubed plus y squared x plus c. And I guess there's a moral to this story, after all.

You know? When I did it the quick way, I made a careless computational mistake earlier in the lecture that sort of threw me off a bit and perplexed me. And it seems it was poetic justice. Because, you know, when I took this method where we pretended we didn't know the answer in advance and just worked the thing out systematically, we came up with the right term. In other words, it should be y squared x, not yx squared. And that was what we saw should be the correct answer.

So maybe there's something to be said about doing things systematically after all. You see? If I can't win them all, at least I have the gift of rationalizing to think that I win them all. Time after time, I snatch defeat from the jaws of victory. No? All right. You know what I mean. Look at it.

Anyway, the question that we've now solved is what? We have worked out this routine. Notice that what we are really doing is the counterpart of calculus of a single variable, that knowing what a partial is, we can integrate with respect to that variable, treating all the other variables as a constant. And this is the general technique. Now let's summarize this more generally.

The general definition is this. And by the way, for the sake of conserving space, I will now abbreviate M(x, y) and N(x, y) just by M and N. But whenever I write M and N here, it's assumed that M and N are functions of x and y. Well what we say is this. M dx plus N dy, which by the way, by a previous definition is always a differential. But this differential is called an exact differential if and only if there exists a function w is f(x, y) such that dw is M dx plus N dy.

Or in terms of our previous discussion, what this means is what? dw is also equal to f sub x dx plus f sub y dy. Since x and y are independent variables, dx and dy are independent variables. So an alternative definition is what? That M dx plus N dy is an exact differential if there exists a function f such that the partial of f with respect to x is M, and the partial of f with respect to y is N.

Now by the way, let's just pause here for a moment. You might suspect that with all the functions to choose from, that no matter how M and N were given, we're bound to find at least one function which satisfies-- I should put this in, because you want both of these conditions fulfilled at the same time. The amazing thing is that it is not always possible to find a function which has a given differential as its total differential. In fact, let me show you why, if we can just do something a little bit tricky here.

Let's suppose that the function f exists such that its partial with respect to x is M and its partial with respect to y is N. Suppose that both of these happen to be differentiable. Take the partial of f sub x with respect to y. That tells me that f sub xy is the partial of M with respect to y. Let's take the partial of f sub y with respect to x. That tells me that the partial of f sub y with respect to x is N sub x. In other words, f sub yx equals N sub x. Now if f is continuous-- see, if we have the right amount of continuity and differentialabiliy that we've talked about in previous lectures, notice that for most well-defined functions-- in particular, if these two partials-- see, we don't know what f is. We're trying to find f. And we're assuming it exists.

The thing that's given in our definition are M and N. So to word this more succinctly, if it turns out that the partial of M with respect to y and the partial of N with respect to x happen to be continuous functions, notice that these two things are equal. And in particular, that means that the partial of M with respect to y has to equal the partial of N with respect to x.

Coming back to the original definition here, this is a rather amazing thing. It says that if this differential is exact, if you take the coefficient of dx-- see, think of M as being the coefficient of dx and N as being the coefficient of dy. If you take the coefficient of dx and differentiate that with respect to y, you must get the same answer as if you took the coefficient of dy and differentiated that with respect to x.

That's an amazing coincidence if that happens. Given two arbitrary functions, this indeed won't happen.

Now before I show you that, let's summarize this in words written down so that I make sure that you understand what I'm saying. What I'm saying is that if the partial of M with respect to y and the partial of N with respect to x are continuous, then if M dx plus N dy is exact, the implication is that the partial of M with respect to y must equal the partial of N with respect to x.

And to invert the emphasis here, another way of saying this is that if the partial of M with respect to y is not equal to the partial of N with respect to x, then M dx plus N dy is not exact. And let me show you that by means of an example. Let's look at the differential-- a very simple-looking one. In fact, it looks far less complicated than the expression that we worked with previously. Let's look simply at y dx minus x dy. In this problem, the coefficient of dx is y. That plays the role of M. The coefficient of dy is minus x. That's what we're calling N, you see. And the partial of M with respect to y is just the partial of y with respect to y, which is 1. The partial of N with respect to x is the partial of minus x with respect to x, which is minus 1.

Since 1 is not equal to minus 1, this is not exact. Because if it were exact, these two partials would have to be equal. What does it mean to say that this is not exact? What it means is that there is not a single function in the whole world that has the property that its partial with respect to x is y, and it's partial with respect to y is minus x. You can look forever. And not only won't you find one, there isn't one. And by the way, let me make one very quick aside here. Notice the difference between saying you can't find one, and there isn't one. There may be one, and you're just not lucky enough to find it.

I'm saying that the reason you don't find an answer here unless you make a mistake and think you've found an answer-- the reason you don't find an answer is there is none. And how can I show you that there is none? Well the best way is to do the same thing that we did in the previous example, and see something go wrong.

Let's suppose there were an answer to this. Well if the partial of f with respect to x equals y, let me integrate this with respect to x, treating y as a constant. It means that the function that I'm looking for must have what form? It's x times y plus some function of y alone. And now we say, gee, we're in pretty good shape. All we've got to do is find what g(y) is, and we're home free. Let's continue to mimic what we did before.

Knowing this, I can take the partial of this with respect to y, OK? Which is what? If I take the partial of this with respect to y, this is f sub y is x plus g prime of y. All right? Now this is the partial of f with respect to y computed from here. I know by hypothesis that the function I'm looking for must have its partial with respect to y equal to this. Consequently, these two expressions must be equal.

And equating these two expressions says that g prime of y-- see, x plus g prime of y must equal minus x. Transposing says that g prime of y is minus 2x. And this is a contradiction. Because look at it-- g prime of y is a function of y alone. And here we have it equal to some function of x. Or another way of looking at it, this says that x depends on y. x is some function of y, which is a contradiction since we assumed that x and y are independent variables.

By the way, just as a quick aside, notice that in the case where we were able to succeed at this-- namely the previous example, x squared plus y dx plus 2 xy dy, M was x squared plus y squared, N was 2 xy. Notice that the partial of M with respect to y is 2y. And the partial of N with respect to x is also 2y.

So it seems that the deciding factor really seems to be that the partial of M with respect to y has to equal the partial of N with respect to x. And the major result is-- and I didn't write this in, because I didn't want to use up too much space. But this is emphasized in the text and in the exercises. If M, N-- M sub y and N sub x all exist and are continuous, then not only can we say that M dx plus N dy is exact implies this part. But we can do the converse too, that in particular, if M sub y equals N sub x, we can conclude that this is exact.

And the reason for this-- and I'll go through this very quickly because the proof is given in the text. I just want to show you the highlights of what happens is, what really happened when we tried to construct f that made things work in one case but not in the other?

I think the best way to do this is to work abstractly here without specifying what M and N are, and see what happens. What we were trying to solve was a pair of equations-- we were trying to find f such that the partial of f with respect to x is M, partial of f with respect to y is N. The first thing that we did was we treated y as a constant and integrated this with respect to x. Well, I can't do this specifically, because I don't know what M is. So let me just put the integral sign in here to say what? I'm integrating this with respect to x, treating y as a constant. That's some function of x and y, you see over here. See M is a function of x and y. I'm holding y constant. I'm integrating this. Plus a constant of integration which depends only on y. All right? Same as before.

What was my next step? I took the partial of this with respect to y, and I was going to compare that with what I knew the partial of f with respect to y had to be-- namely, N. So I then took what? The partial of this with respect to y. That gave me the partial of this integral with respect to y plus g prime of y. And equating that to N, I get that g prime of y was N minus the partial with respect to y integral M dx.

And again, don't be alarmed by this expression. If M had been given explicitly, I would have solved explicitly for what this function was. The key point is to notice that this side here is independent of x. Consequently, this equation will be a contradiction unless the right-hand side is independent of x.

In other words, we have determined f up to this function g(y). And what's going to happen is we are going to be able to compute g(y) if this expression is independent of x, namely just by integrating this. But if g(y) is not independent of x, we are in trouble. It means that it's not going to exist. The key step is, is this independent of x? And the answer is, the best way to find out in terms of calculus is to take its derivative with respect to x. If this depends on y alone, its partial with respect to x must be 0.

In other words, if this partial with respect to x is not 0, it means that this expression varies with x. At any rate, to see what this derivative is, we just differentiate term by term. And then we say, you know? It's rather interesting. If this thing had been first-- notice that if you integrate with respect to x and then differentiate with respect to x, you wind up with just the integrand.

Well again, if we have enough continuity, the order of differentiation is irrelevant. We can reverse the order without changing anything if we now interchange this order, take the partial first with respect to x. That will give us just M. And the partial of M with respect to y is just the partial of M with respect for y, of course. What we're saying is that from this step, we get to this step. And this in turn implies this. Notice that the only way that this can be 0 is if the partial of N with respect to x equals the partial of M with respect to y.

So in summary then, this is a very interesting device. It tells us how to tell whether a differential is exact. And the proof is very nice in this case because the proof exactly imitates how we construct the function f if it turns out to be exact. But the point is, if given M dx plus N dy, if the partial of M with respect to y is not equal to the partial of N with respect to x, there's no hope that the function will be exact-- the differential be exact.

At any rate, we will have sufficient exercises for drill in how one uses exact differentials. This concludes block three of our material. Our next block of material which will concern linear systems of equations will begin with our next lecture. And until next time, goodbye.

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